Rationality of Exercising and Valuating Options of Variable Annuities

Riley Jones*

The University of North Carolina at Charlotte

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Abstract

A variable annuity is a type of savings policy typically offered by insurance companies. The policyholder makes an upfront payment to the insurance company and, in return, the insurance company is required to make a series of payments starting at an agreed upon date. For a higher premium, many insurance companies offer additional guarantees or options which protect policyholders from various market risks. This research is centered around creating a pricing framework for two of these options: the guaranteed minimum income benefit and the reset option. This framework is used to explore how various parameters and assumptions affect the pricing of these options. Additionally, a critical value for future interest rates is calculated to determine the rationality of exercising the reset option. This will be able to provide insight to both the policyholder and policy writer on how their future projections on the performance of the stock market and interest rates should guide their respective actions of exercising and pricing variable annuity options. This can help provide details into the value of adding options to a variable annuity for companies that are looking to make variable annuity policies more attractive in a competitive market.

Keywords: annuity, option, pricing, Monte-Carlo, interest rate

*Correspondence to rjone157@uncc.edu
1 Introduction

A variable annuity is a long-term, tax-deferred product, whose funds are equity-linked from the time of the initial payment until the annuitization date (the accumulation period). The initial payment is invested into sub-accounts made up of mutual funds and other investments. The growth of the investments during the accumulation phase affects the payout of the annuity at the annuitization date (often at retirement). This product is designed to provide post-retirement income.

While this product is targeted at providing financial security throughout retirement, there is a large amount of risk inherent. This risk stems largely from the performance of the markets from which the value of the annuity is derived. If the markets perform poorly over the accumulation period, an individual could have a post-retirement income significantly less than expected. With retirement being something that few people are willing to risk, it is important to be able to offer something that reduces the risk of the variable annuity. The most common way to protect the annuity balance from poor investment performance is the inclusion of a guaranteed minimum benefit when the contact is underwritten.

In fact, when insurance companies began to include guaranteed minimum benefits in their variable annuity products in the late 1990’s, there was a large growth in the number of polices sold [5]. This made variable annuities a more attractive option because it reduced the level of risk in these policies to policyholders. Today, guaranteed minimum benefit options are very common with variable annuities. According to Drexler [5] in 2016, 76% of policyholders chose to purchase a guarantee when the option was available with their variable annuity. With a decline in the number of pension plans and other traditional forms of retirement plans, many people are looking into less traditional ways to be financially secure through retirement. Since variable annuities are a long-term investment which can have a very low risk (with a guaranteed minimum benefit), they are a great option for retirement.

Guaranteed minimum benefits can come in many forms; however, the four main types are Guaranteed Minimum Withdraw Benefits (GMWB), Guaranteed Minimum Death Benefits (GMDB), Guaranteed Minimum Accumulation Benefits (GMAB), and Guaranteed Minimum Income Benefits (GMIB). While Bauer et al. (2008) [3] and Bacinello et al. (2011) [2] have created a valuation and pricing framework for guaranteed minimum benefits in general, there is little research into Guaranteed Minimum Income Benefits. This is because variable annuities can be made into very dynamic policies when a variety of additional guarantees are included. Furthermore, while companies have advanced pricing tools and methods to price the cost of these policies, current research [9] indicates that these policies are typically underpriced. This research will specifically study variable annuities with a GMIB and a reset option.

A GMIB could have many additional options added to it; however, it essentially provides the policyholder with two options at the annuitization date: annuitize the accumulated value of investments at prevailing rates or annuitize a guaranteed amount at a set rate \( g \) (determined at the onset of the contract). The reset option, as defined within this paper, provides a third option which allows the policyholder to defer the annuitization date to a
later time. This is useful if the policyholder is not in need of a payment at the annuitization date and thinks the policy will gain value over the next year.

In this paper, the pricing of a GMIB is constructed to find fair values of the guaranteed annual payment rate \( g \) given different levels of a fee rate \( c \). Additionally, the reset option is analyzed to determine critical values for future interest rates which will determine the rationality of exercising the reset option. It is important to note that the reset option is not considered in the pricing of the GMIB.

2 Framework

A Monte-Carlo simulation approach is used in the pricing of the GMIB and for the calculation of the critical interest rate values. This allows for more specialization of the model parameters in place of using a closed-form formula approach. When looking at the payoff of the GMIB, it can be expressed by the maximum of the value of the benefit base and the investment account as given by:

\[ P(T) = \max[BB(T), S_f(T)] \]  

(1)

where \( T \) indicates the terminal time (annuitization date), \( BB(T) \) represents the value of the benefit base and \( S_f(T) \) represents the investment account with all fees deducted. Throughout this paper, a value of \( T = 20 \) will be used, indicating a 20-year accumulation period, and fees will be deducted annually. The payoff will be the maximum of the two values because the policyholder will want to maximize the value of their payments. The value of the benefit base at annuitization can be expressed as:

\[ BB(T) = S(0)(1 + r_g)^T \cdot g \cdot a_{20}(T) \]  

(2)

where:

- \( S(0) \) is the amount of the initial premium.
- \( r_g \) is the guaranteed annual rate. A rate between 4% - 6% is industry standard. A rate of 5% will be assumed for the entirety of this paper.
- \( a_{20}(T) \) is the market value of a twenty-year annuity with payments of $1 beginning at time \( T \).
- \( g \) is the guaranteed payment rate specified at the beginning of the contract. Note that if \( g \) is priced fairly, then \( g \) should be the multiplicative inverse of \( a_{20}(T) \); however, this relationship is affected by the prevailing interest rate at time \( T \). Since the interest rate at time \( T \) is not known (in practice) at any time before \( T \), its value has to be approximated. It is often the case that \( g \) is set so conservatively that \( g \cdot a_{20}(T) < 1 \).

For the investment account, insurance companies typically deduct an annual fee that is a percentage of the benefit base. The amount deducted from the investment account every year can be expressed by:

\[ f_1(n) = cS(0)(1 + r_g)^n, \quad n = 1, 2, \ldots, T \]  

(3)
With this, the fee that will be deducted every year is known at the onset of the contract. However, another method for calculating the amount of the annual fee can be given by:

$$f_2(n) = f_1(n) \times g \times a_{20}(T), \quad n = 1, 2, \ldots, T$$

(4)

With this equation for an annual fee, two extra terms are included: $g$ and $a_{20}(T)$. While these terms should multiply to equal one, resulting in $f_2(n) = f_1(n)$, this is often not the case. The value of $a_{20}(T)$ will change as the future expectation of the interest rate at time $T$ changes. The use of these two fee amounts and the effects they have on the pricing of the GMIB is discussed in the Results section. However, $f_1(n)$ is industry standard.

Since the value of the investment account is strongly correlated with the performance of the stock market [5], a Geometric Brownian Motion is used to simulate future paths of the investment account. Thus, the value of the investment account (before fees) is given by:

$$S(t) = S(0)e^{(r - \sigma^2/2)t + \sigma W(t)}$$

(5)

where:

- $r$ is the interest rate. Note that in this model the interest rate is assumed to be constant. However, this is not the best approximation for future interest rates. More precise models consider $r(t)$ to be a stochastic process. For simplicity, $r$ is held constant in this paper.

- $\sigma$ is the market volatility. This is also assumed to be a constant value.

- $W(t)$ is a standard Brownian Motion

From this equation we can generate numerous realizations of the investment account path. Figure 1 shows 100 realizations of the investment account path given $S(0) = 100,000$, $\sigma = 10\%$ and $r = 5\%$:
It is important to note that each path is not one continuous Geometric Brownian Motion from time 0 to \( T = 20 \). Instead, each realization is a combination of 20 Geometric Brownian Motions of length one year. Each path is stopped at the end of the year and the fee is deducted. Then the motion continues but starting at the new account value. Figure 2 is a close-up chart of one realization of the investment account. The gap represents the fee amount that was deducted from the investment account, in this case \( f_1(10) \).

Now that we have an expression for the benefit base and the investment account (adjusted for fees), we can calculate the value of the GMIB. This value can be expressed by:

\[
V(g) = \mathbb{E}[(1 + r)^{-T}P(T)]
\]  

This \( V(g) \) value holds all other parameters constant and varies the level of \( g \). While one simulation would produce a value for \( V(g) \), this value is not reliable. A better estimation for \( V(g) \) would be the mean of many more iterations. The most robust model run in this paper considers 200,000 values of \( V(g) \) for each small increment of \( g \).

The fair guaranteed annuity payment rate is the value of \( g = g^* \) such that:

\[
V(g^*) = S(0)
\]  

If \( V(g) > S(0) \), then the insurance company is undercharging for the GMIB. Likewise, if \( V(g) < S(0) \), then the insurance company is overcharging for the GMIB.
For the reset option, the same modeling process is used. The only difference is that the model is extended by 1-year to include the projections from time $T$ to $T + 1$. This option can be expressed by setting the new terminal time to $T' = T + 1$. While the interest rate is at the same level from time 0 to $T$, its value is changed for the year following the initial annuitization date, but still at a constant level. This allows for results on how the change in the interest rate will affect the values of the benefit base and the investment account. The interest rate which results in a benefit base value equal to the investment account at time $T'$ will be referred to as the critical interest rate value $r^*$ from $T$ to $T'$. If $S_f(T) > BB(T)$, then the reset option should be exercised if the policyholder expects future interest rates to be below $r^*$ (and vice versa). Further details are given in the Results section.

There are several additional points and assumptions to note for this model:

- One of the largest assumptions in this model is that the interest rate is constant. A more robust model would consider interest rates to be a stochastic process.

- Volatility is also assumed to be a constant value throughout this paper. While this assumption does not have as large of an impact as interest rates being constant, it is important to note that a more robust model would also have volatility following a stochastic process.

- The value of $g$ has a significant impact on the value of the GMIB. This value is set by the insurance company at the onset of the policy. Many insurance companies set this value so conservatively that even if $S(0)(1 + r_g)^T$ is larger than the investment account at the annuitization date, the policyholder still may be better off annuitizing the investment account. That is why this model defines the benefit base as $BB(t) = S(0)(1 + r_g)^t \cdot g \cdot a_{20}(T)$.

- In practice, there may be factors which result in the policyholder breaking the contract prior to the annuitization date. When this happens, it is called a lapse. The risk of lapse for the GMIB is not included in its pricing.

- While some guaranteed minimum benefits have a death benefit or expire upon the death of the policyholder, mortality risk is not considered in this model. The policy would be passed on to a beneficiary if the policyholder passes away prior to the termination of the policy.

- While some companies make annuity payments monthly, payments for the annuities priced in this paper will paid annually for 20 years. There is flexibility for the term of the policy; however, 20 years is selected because the average life span after retirement at age 65 is 19.3 years for males and 21.7 years for females in the United States [11].

3 Results

In this section, the value of the GMIB is given as a function of the guaranteed annual payment rate $g$ to determine the fair rate $g^*$ for varying fee rates $c$ and critical values for
the reset option are given. Additionally, discussion is provided into other parameters of the model such as the volatility of the market, fee structure implemented, and the number of iterations run. Unless otherwise stated, the fee structure follows that of $f_1(n)$ and the following parameters are used in the model: $S(0) = $100,000, $T = 20$, $\sigma = 10\%$, $r = 5\%$ and $r_g = 5\%$.

### 3.1 GMIB Option

#### 3.1.1 Probability of Exercising the Benefit Base

An important point of consideration when offering a guaranteed minimum benefit is understanding the probability that the benefit will be exercised. If there is a very high likelihood that the benefit will be exercised, then the premium will be higher and its risk should be hedged appropriately. Figure 3 shows the probability that the GMIB will be exercised for different levels of $c$ and $g$. The value of $g$ is given between 5% and 10% because these values correspond with a realistic range of future interest rates from 0% to 9%. If interest rates were to exceed 9%, then a value of $g$ higher than 10% should be explored. Also, the fee rate ranges from .5% to 1% because this is current industry standard [1]. Note how the data appears to converge to a certain function as the number of iterations is increased.

![Figure 3: Probability of Exercising the Benefit Base](image)

(a) 10,000 iterations  
(b) 100,000 iterations  
(c) 200,000 iterations
These graphs are generated from points calculated for given levels of $g$ and $c$. They are connected by lines at each level of $c$ to show a trend. For each level of $c$ there are 51 values of $g$ generated on equidistant intervals from .05 to .10. The number of iterations listed is how many realizations of the investment account and the benefit base were calculated for each level of $c$ and $g$. The total number of realizations in which the value of the benefit base was greater than that of the investment account at time $T$ divided by the number of iterations is how the probability of exercising the benefit base was calculated. Each chart has 306 points calculated.

From this graph we can see that both $g$ and $c$ have a positive relationship with the probability of exercising the benefit base. As both $c$ and $g$ increase, the probability that the benefit base is exercised increases. However, within the range of current industry standard, we can see that $g$ has a much larger impact on the probability of exercising the benefit base than $c$ does. A change from $g = .05$ to $g = .1$ results in a change of approximately 50 percentage points while the change from $c = .5\%$ to $c = 1\%$ results in a change of approximately less than 10 percentage points.

While simulations were run for only 51 points of $g$ for each given level of $c$, the points appear to follow the trend of a function. In order to find the probability of exercising the benefit base for any level of $g$ given a set value of $c$ at .5\%, .6\%, .7\%, .8\%, .9\% or 1\%, a regression analysis can be run for each of these $c$ values. While a simple linear regression can be fit to find a relationship between $g$ and the probability of exercising the benefit base, a curve appears to be a better fit for the data.

For example, Figure 4a shows the data for $g$ given $c = 1\%$. A line of best fit is displayed in Figure 4b and a second degree polynomial fit to the data is displayed in Figure 4c. Visually, we can see that the second degree polynomial is a better fit to the data than a linear function. This is reinforced by the adjusted $R^2$ values for both functions. The adjusted $R^2$ value of the linear function is 0.9711 and the adjusted $R^2$ value of the second degree polynomial is 0.9999. The closer the $R^2$ value is to 1, the more variation in the probability of exercising the benefit base is explained by the model. Thus, the second degree polynomial is a very good fit to the data. To show that the second degree polynomial is a significantly better fit to the data than the linear function, an analysis of variance (ANOVA) test was run between the models. The results are given in the following table:

<table>
<thead>
<tr>
<th>Analysis of Variance Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1: $y \sim x$</td>
</tr>
<tr>
<td>Model 2: $y \sim \text{poly}(x, 2, \text{raw}=\text{TRUE})$</td>
</tr>
<tr>
<td>Res.Df</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Signif. codes: 0 ‘<em><strong>’ 0.001 ‘</strong>’ 0.01 ‘</em>’ 0.05 ‘.’ 0.1 ‘ ’ 1</td>
</tr>
</tbody>
</table>

The linear model is given as "Model 1" and the second degree polynomial model is given as "Model 2". Since there is a significant p-value, we reject the null hypothesis that one
model does not fit better than the other model. Thus indicating that the second degree polynomial is a better fit for the data. Since the adjusted $R^2$ is very high at .9999, there is no need to explore higher degree polynomials to fit the data.

We have now concluded that a second degree polynomial is a very good fit for the $g$ data given $c = 1\%$. Repeating the same process for the other five levels of $c$ has the same results that a second degree polynomial is a better fit to the data than a linear function and has an adjusted $R^2$ value of .9999. With the following equations in Table 1, we can now find a predicted probability of exercising the benefit base for any level of $g$ given a fixed fee rate of either .5\%, .6\%, .7\%, .8\%, .9\% or 1\%.

The ability to generalize the data to a function is significant because we can now predict the probability of exercising the benefit base for any level of $g$. This makes the results more useful to an insurance company who may want to calculate the probability of exercising the benefit for a specific level of $g$ that was not particularly tested for in the model. Now that the effect $g$ and $c$ have on the probability of exercising the benefit base is given, we can now shift focus to determining what the fair value of $g$ will be.
Table 1: Estimated Probability of Exercising the Benefit Base

<table>
<thead>
<tr>
<th>c</th>
<th>Regression Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5%</td>
<td>$-10976.24g^2 + 2699.51g - 76.88$</td>
</tr>
<tr>
<td>0.6%</td>
<td>$-11150.96g^2 + 2700.14g - 74.24$</td>
</tr>
<tr>
<td>0.7%</td>
<td>$-11498.39g^2 + 2726.93g - 72.60$</td>
</tr>
<tr>
<td>0.8%</td>
<td>$-11601.42g^2 + 2713.42g - 69.42$</td>
</tr>
<tr>
<td>0.9%</td>
<td>$-11869.98g^2 + 2726.67g - 67.29$</td>
</tr>
<tr>
<td>1%</td>
<td>$-11890.48g^2 + 2701.61g - 63.78$</td>
</tr>
</tbody>
</table>

### 3.1.2 $V(g)$ and Fair Values of $g$

It is important for both the insurance company and the policyholder to know what the fair value of $g$ is when signing the policy. Using the same data that was computed in the previous section, the fair value of $g$ is now calculated for the same six fee rates varying from .5% to 1% on equidistant intervals. Recall that the fair value of $g$ is the $g^*$ such that $V(g^*) = S(0)$. A $V(g)$ value is calculated with each iteration; however, this value is not a very reliable estimate. So, the $V(g)$ values presented in Figure 5 for each level of $g$ and $c$ is the mean of all of the iterations. This is done for the same 306 combinations of $g$ and $c$ calculated in the previous section. The values are then connected by lines by the fee value to show the trend of the data. The most critical point of consideration in this section is the initial value of the investment account (100,000). When $V(g)$ is equal to 100,000, then the GMIB option is fairly priced. This is seen by the intersection of each line with the grey dotted line. If $V(g)$ is greater than 100,000, then the insurance company is undercharging for the option. On the other hand, if $V(g)$ is less than 100,000, then the insurance company is overcharging for the GMIB.

As seen in Figure 5, $g$ has a positive relationship with the value of $V(g)$ and the fee charged $c$ has a negative relationship with $V(g)$. As the $g$ value increases, $V(g)$ increases which makes the GMIB more valuable. However, as the fee rate charged increases, $V(g)$ decreases making the value of the GMIB less valuable. This makes sense because a contract with a higher fee is less valuable to the policyholder.

It is important to note that none of the combinations of $g$ and $c$ calculated resulted in a $V(g)$ value of exactly 100,000. However, the strict monotonicity apparent in the chart implies that the function in injective. Hence, there exists some value of $g$ for each level of $c$ such that $V(g)$ equals to 100,000. This $g$ level is calculated by taking the two levels of $V(g)$ closest to 100,000 (both above and below) and calculating the proportion of the difference in the values that 100,000 is from the smaller $V(g)$ value. This proportion is then given out of .001 and added to the $g$ corresponding with the smaller $V(g)$. This gives the best estimate of $g^*$ presented in Table 2 for each level of $c$.

These fair values of $g$ are best estimates given the parameters of the model. If the parameters of the model were to change, then these values would change as well. Additionally,
there is a degree of uncertainty with each fair value given. The degree of uncertainty varies with the number of iterations run in the model. If only a few number of iterations were run, then each fair value of $g$ would have a very large variance level associated with it. Since 200,000 iterations were run at each point, the fair $g$ values presented in Table 2 have a relatively smaller variance level. However, future work should be done into calculating the exact certainty of these values.

Table 2: Fair values of $g$

<table>
<thead>
<tr>
<th>$c$</th>
<th>$g^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.0558</td>
</tr>
<tr>
<td>0.006</td>
<td>0.0581</td>
</tr>
<tr>
<td>0.007</td>
<td>0.0601</td>
</tr>
<tr>
<td>0.008</td>
<td>0.0619</td>
</tr>
<tr>
<td>0.009</td>
<td>0.0633</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0645</td>
</tr>
</tbody>
</table>
3.1.3 Fees given by $f_1(n)$ vs. $f_2(n)$

As stated at the beginning of this section, the fee structure used in sections 3.1.1 and 3.1.2 follows that of $f_1(n)$. Now, the fee structures of $f_1(n)$ and $f_2(n)$ are compared to find the probability of exercising the benefit base and the fair level of $g$. While figures with less than 200,000 iterations were used in previous sections, all models in this section use 200,000 iterations.

The relationship between $f_1(n)$ and $f_2(n)$ can be seen as $f_2(n) = f_1(n) \ast g \ast a_{20}(T)$. Since the interest rate is assumed to be constant at 5% from 0 to $T$, the value of $a_{20}(T)$ will be constant as well. So the effect $a_{20}(T)$ has on the total fee deducted is the same for any level of $g$ and $c$. In a model where interest rates vary, $a_{20}(T)$ would vary as well. This would make its effect on the fee structure more complex. Figure 6 shows the probability of exercising the benefit base for each fee structure. While the results follow similar structures, we can see that $f_1(n)$ has more variance in $c$ at a lower level of $g$ and converges to a level of less variance in $c$ at a higher level of $g$. Meanwhile, the variance in $c$ for $f_2(n)$ appears to be constant for all $g$. Additionally, $f_1(n)$ has less change in the probability of exercising the benefit base as $g$ increases. For example, for $c = 1\%$, the probability of exercising the benefit base changes from 41$\%$ to 88$\%$ (47 percentage points) as $g$ goes from .05 to .1 under $f_1(n)$. Under $f_2(n)$, the probability of exercising the benefit base changes from 34$\%$ to 90$\%$ (56 percentage points) as $g$ goes from .05 to .1.

![Figure 6: Probability of Exercising the Benefit Base and Fee Structure](image)

(a) $f_1(n)$

(b) $f_2(n)$

The fee structure also has a significant impact on the value of $V(g)$. As seen in Figure 7, there is an upward shift in the $V(g)$ values from $f_1(n)$ to $f_2(n)$. Additionally, the impact that $c$ has on $V(g)$ is less in $f_2(n)$ than $f_1(n)$ because the values are much closer together under $f_2(n)$. This results in fair values of $g$ under $f_2(n)$ which are much lower than the fair values calculated under $f_1(n)$. This makes sense because $g$ is a factor in $f_2(n)$ but not in $f_1(n)$. If the additional $g$ term is lower, then the amount being deducted from the investment account annually will be less. This will make the option more valuable, which translates into a higher $V(g)$ value. This is seen by the upward shift between the graphs.
3.1.4 The Effect of Volatility on the Model

Another assumption held to be true in the previous sections is that the volatility is a constant at 10%. So, how does changing this assumption affect the overall results? In order to test this, the model was run at a volatility level of 2%, 10% and 20%. The probability of exercising the benefit base at each of these volatility levels is given in Figure A.1. In this, we can see that the volatility has a very significant impact on the probability of exercising the benefit base. With a low level of volatility, the data appears to fit a logarithmic distribution and varies from 0% to 100%. As the volatility increases, the probability of exercising the benefit base is contained in a smaller interval and the data follows more of a quadratic (almost linear) function. Additionally, the lines are less smooth with increased volatility which indicates more variance in the individual results.

Similarly, Figure A.2 shows the value of $V(g)$ at each volatility level. The fair value of $g$ varies significantly based on different volatility levels. At a low volatility level of 2%, the fair value of $g$ is nearly the same for every value of $c$ since the $V(g)$ functions converge to a line. At a level of 10%, there is more variation amongst $c$ values which results in different fair values of $g$. Lastly, at a volatility level of 20%, there is no fair $g$ value between .05 and .1 for fees between .5% and 1%. The variation caused by the change in volatility shows that the volatility assumption has a very large effect on the results given in 3.1.1 and 3.1.2.

3.2 Reset Option

3.2.1 $BB(T')$ and $S_f(T')$

In order to assess the rationality of exercising a reset option as defined by the ability to delay the annuitization date, in this case, by one year, simulations are made for both the performance of the benefit base and the investment account for the year following the initial annuitization date. The performance of these accounts is considered under a new interest rate value (which is the growth rate for the investment account). This value is still constant.
but varies from 0% to 10%. Figure 8 shows the projected values for the benefit base and the investment account at time $T'$ at different interest rate levels.

Figure 8: Values of $BB(T')$ and $S_f(T')$

Each graph is composed of 100 equidistant intervals (101 points) connected by lines to show the trend of the accounts. Each point for the investment account is the mean value of the total number of realizations calculated. The benefit base rolls-up to a fixed level, so iterations are not run for its value.

It can be seen that there is a certain interest rate value for the year following the initial annuitization date such that the value of the benefit base equals that of the investment account. This value is defined earlier as the critical interest rate value $r^*$. It is with this value that the rationality of exercising the reset option (at time $T$) is determined. For example, in Figure 8c the values of of the future benefit base and investment account at time $T'$ are given for an assumed level of $g = .065$ and $c = .007$. If the policyholder was in a situation where they had more money in their investment account than their benefit base at time $T$, then they should rationally exercise the reset option if they expect future interest rates to
drop below \( r^* = 4.35\% \). This is because if interest rates drop below this level for the following year, the benefit base is now expected to have more value than the investment account. Likewise, if at time \( T \) the benefit base has more value than the investment account, the reset option should be rationally exercised if the policyholder expects interest rates to be above 4.35%.

This can be more generally stated as if \( S_f(T) > BB(T) \), then the reset option should be exercised if the policyholder expects future interest rates to be below \( r^* \). Likewise, if \( S_f(T) < BB(T) \), then the reset option should be exercised if the policyholder expects future interest rates to be above \( r^* \). The critical interest rate value stated above of 4.35% was specifically for \( c = .007 \) and \( g = .065 \). What value this critical interest rate has for different levels of \( c \) and \( g \) is given in the next section.

### 3.2.2 Critical Interest Rate Values

To plot the critical interest rate value for different levels of \( c \) and \( g \), the data for the chart shown in the previous section is replicated for 306 different combinations of \( c \) and \( g \). The critical interest rate value \( r^* \) is taken from each of these calculations and placed onto a graph as seen in Figure 9. The number of iterations that went into calculating the future investment account value at each interest rate level within the individual 306 combinations is 200,000. This method was computationally heavy and required a lot of computing power and time.

![Figure 9: Critical values for the interest rate from \( T \) to \( T' \) (\( r^* \))](image)

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In a similar fashion to the work done earlier, a regression function can be run so that the critical interest rate value can be calculated for any value of \( g \). While a linear function could be fit to the data, a second degree polynomial function is a better fit for each level of \( c \). A statistically significant p-value from an ANOVA test verifies this. Additionally, a very high adjusted \( R^2 \) value tells us that it is not necessary to explore fits for functions with higher degree polynomials. The regression functions are given in Table 3.

Table 3: Estimated Value of Critical Interest Rate

<table>
<thead>
<tr>
<th>( c )</th>
<th>Regression Function</th>
<th>Adjusted ( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5%</td>
<td>(-5.50g^2 + 2.46g - 0.0953)</td>
<td>0.9996</td>
</tr>
<tr>
<td>0.6%</td>
<td>(-5.58g^2 + 2.49g - 0.0943)</td>
<td>0.9997</td>
</tr>
<tr>
<td>0.7%</td>
<td>(-5.96g^2 + 2.56g - 0.0944)</td>
<td>0.9997</td>
</tr>
<tr>
<td>0.8%</td>
<td>(-5.64g^2 + 2.54g - 0.0916)</td>
<td>0.9997</td>
</tr>
<tr>
<td>0.9%</td>
<td>(-5.51g^2 + 2.55g - 0.0898)</td>
<td>0.9997</td>
</tr>
<tr>
<td>1%</td>
<td>(-5.65g^2 + 2.59g - 0.0889)</td>
<td>0.9997</td>
</tr>
</tbody>
</table>

3.2.3 Yield Curve and Future Expectation of Interest Rates

What the policyholder expects the interest rate to be from time \( T \) to \( T' \) is very subjective. An individual could have a personal opinion that the Federal Reserve is going to raise the targeted federal funds rate; however, this is based entirely off of intuition. Thus, it is very hard to predict the policyholder’s expectation. However, there are few metrics which reflect the general market sentiment on the stability of future interest rates and the direction of the economy as a whole. One in particular is the Treasury yield curve.

The Treasury yield curve reflects the short, intermediate, and long-term rates of U.S. Treasury securities. The shape it takes reflects the current market sentiment on the direction of the economy and interest rates. For example, the yield curve from March 2nd, 2009 in Figure 10 shows a generally normal yield curve. Its value starts low and curves upward. The short-term bonds have lower yields to reflect the fact that the investor’s money is more secure. This type of curve is common when the economy is stable and growing at a healthy pace. If this is the shape of the curve at time \( T \), the policyholder may be more inclined to have an expectation that interest rates will remain the same or be raised.

When the yield curve begins to flatten out or have a hump as seen by the yield curves from March 1st, 2018 and April 12th, 2019 in Figure 10, this typically indicates a period of economic slowdown. When the economy is slowing down, the policyholder may be more likely to have the expectation that interest rates will be lower. Also, when a yield curve is flat, this may indicate that the yield curve is in transition to becoming inverted. An example of an inverted yield curve can be seen by the yield curve on March 1st, 2007 in Figure 10. For this curve, there are lower interest rates as you move along the curve. This reflects the current sentiment that now is the last chance for investors to lock in interest rates before the rates fall. This type of curve points strongly in the direction of an economic slowdown.
While this curve is rare, it is important to note that its occurrence will most likely result in informed policyholders having a lower expectation for future interest rates. This can have a significant impact on the exercise rate of the reset option.

![Yield Curve](image)

Figure 10: U.S. Treasury Yield Curves [13]

4 Conclusion

In conclusion, results have been given to determine the probability of exercising the benefit base, the fair value of $g$, and the rationality of exercising the benefit base. Regression functions were produced to determine the probability of exercising the benefit base for any value of $g$. Critical $g$ values for fee levels of .5%, .6%, .7%, .8%, .9% and 1% were given to be .0558, .0581, .0601, .0619, .0633, and .0645, respectively. Discussion was provided into how the fee structure and the volatility affect the results of the model. Lastly, regression functions were produced to determine the critical interest rate value for any value of $g$. These critical interest rate values can be used to determine the rationality of exercising the reset option.

5 Future Work

Future work should be done into calculating and improving the confidence levels of these results. Additionally, this model can be made more robust be introducing stochastic processes for the interest rate and for the volatility. There is also opportunity for interesting work into the optimal annuitization date, where the annuitization may occur at any point during a period of time instead of at a fixed point.
6 Acknowledgements

A special thanks to Dr. Adriana Ocejo for her continuous support throughout the entirety of this work and to Dr. Wafaa Shaban and Dr. Ted Amato for their review and feedback on this project.

References


A Figures

Figure A.1: Probability of Exercising the Benefit Base with Volatility
Figure A.2: $V(g)$ with Volatility

(a) $\sigma = 2\%$

(b) $\sigma = 10\%$

(c) $\sigma = 20\%$